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# ON BCL-ALGEBRA

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## ABSTRACT:

It has been found that the BCL-algebra is more extensive class than BCK/BCI/BCH-algebra. In this paper we study some properties of BCL-algebra of type $(2,0)$. We also find deformation of such algebra and illustrate the connection between divisible algebra and deformation function.

## KEYWORDS:

BCL-algebra; d-algebra; BCH-algebra; BCI-algebra; BCK-algebra; deformation

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# 1. INTRODUCTION

A new class of algebra of type $(2,0)$ called BCL-algebra is presented in [1]. Liushowed in [1, Theorem 2.4] that a proper BCL-algebra does exist, if such BCL-algebra is not BCK/BCI/BCH-algebra. It also has been shown in [1, Theorem 2.1] that any BCK/BCI/BCH-algebra is a BCL-algebra. The aim of this paper is to find when the converse of Theorem 2.1 in [1] is true. That is, to show when a BCL-algebra could be a BCK/BCI/BCH-algebra. The case where a BCL-algebra can be a BCH-algebra is studied and given in [1, Theorem 2.2]. Later in the paper we study deformation of BCL-algebra. The work in this part is motivated by the results in [3] on deformations of d/BCK-algebra.

We start in Section 2 by introducing the notions of BCL/d/BCH/BCI/ BCK-algebra respectively. Then, in Section 3, we investigate the relation between BCL-algebra and d/BCH/BCI and BCK-algebra. We give examples throughout the paper. The main results in this section are given in Theorem 3.1 which shows that a d-algebra $X$ satisfying $(x*y)*z=(x*z)*y$ for any $x,y,z \in X$ is a BCL-algebra, Theorem 3.2 and Theorem 3.5 which gives the sufficient conditions which make a BCL-algebra become a BCK/BCI-algebra. In the final section of this paper , we define deformation function, deformation point and divisible algebra.We are concerned on the deformationof BCL-algebra. The main results in this section is Proposition 4.1 which gives a deformation of BCL-algebra and Theorem 4.1 that illustrate the connection between divisible BCL-algebra and a given map defined using associators of a non-zero element in $X$.

# 2. PRELIMINARIES

We give here the definitions of BCL/d/BCH/BCI/BCK-algebra from [1,2,3]. We refer the reader to [4] and [5] for further information on BCI/BCK -algebra.

**Definition 2.1:** [1, Definition 2.1] An algebra $(X;*,0)$ of type $(2,0)$ is a BCL-algebra if it satisfies the following conditions for any $x,y,z \in X$:

1) BCL-1: $x*x=0;$
2) BCL-2: $x*y=0$ and $y*x=0$ imply $x=y;$
3) BCL-3: $((x*y)*z)*((x*z)*y)*((z*y)*x)=0.$

**Definition 2.2:** [2, p2] An algebra $(X;*,0)$ of type $(2,0)$ is a d-algebra if it satisfies the following conditions for any $x,y \in X$:

1) d-1: $x*x=0;$
2) d-2: $0*x=0;$
3) d-3: $x*y=0$ and $y*x=0$ imply $x=y.$

**Definition 2.3:** [1, Definition 1.3] An algebra $(X;*,0)$ of type $(2,0)$ is a BCH-algebra if it satisfies the following conditions for any $x,y,z \in X$:

1) BCH-1: $x*x=0$;
2) BCH-2: $x*y=0$ and $y*x=0$ imply $x=y;$
3) BCH-3: $((x*y)*z)*((x*z)*y)=0.$

**Definition 2.4:** [1, Definition 1.1] An algebra $(X;*,0)$ of type $(2,0)$ is a BCI-algebra if it satisfies the following conditions for any $x,y,z \in X$:

1) BCI-1: $x*x=0;$
2) BCI-2: $x*0=0$ imply $x=0$;
3) BCI-3: $x*y=0$ and $y*x=0$ imply $x=y;$

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4) BCI-4: $((x * y) * (x * z)) * (z * y) = 0;$
5) BCI-5: $(x * (x * y)) * y = 0.$

**Definition 2.5:** [3, p316] An algebra $(X;*,0)$ of type $(2,0)$ is a BCK-algebra if it satisfies the following conditions for any $x, y, z \in X$:

1) BCK-1: $x * x = 0;$
2) BCK-2: $0 * x = 0;$
3) BCK-3: $x * y = 0$ and $y * x = 0$ imply $x = y;$
4) BCK-4: $((x * y) * (x * z)) * (z * y) = 0;$
5) BCK-5: $(x * (x * y)) * y = 0.$

## 3. RESULTS ON BCL-ALGEBRAS

In this section, we give some properties related to BCL-algebra. We give necessary conditions for a BCL-algebra to become a d/BCK/BCI/BCH-algebra. We start with the following example of a d-algebra which is not a BCL-algebra.

**Example 3.1:** Let $X := \{0, 1, 2, 3\}$ be a set in which $*$ is defined by the following Cayley table:

| * | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 2 | 0 | 1 |
| 3 | 3 | 1 | 2 | 0 |

We can easily see that $(X;*,0)$ is a d-algebra and that BCL-1 and BCL-2 does hold. For BCL-3, we can see that if $x = 3, y = 2$ and $z = 1$, then

$$((3 * 2) * 1) * ((3 * 1) * 2) * ((1 * 2) * 3) = (2 * 1) * (1 * 2) * (3 * 3) = 2 * 3 * 0 = 1 * 0 = 1 \neq 0.$$

Thus $(X;*,0)$ is not a BCL-algebra.

**Lemma 3.1:** Not every d-algebra is a BCL-algebra.

This leads us to find a sufficient axiom (as shown in the next theorem) if satisfied then the d-algebra will become a BCL-algebra. We will label the extra axiom $(x * y) * z = (x * z) * y$ by d-$4^{+}$ for brevity.

**Theorem 3.1:** A d-algebra $(X;*,0)$ satisfying d-$4^{+}$is a BCL-algebra.

Proof: Let $(X;*,0)$ be a d-algebra. It is clear that BCL-1, BCL-2 are satisfied. We only need to show that BCL-3 is valid. We have $((x * y) * z) * ((x * z) * y) * ((z * y) * x) = 0 * ((z * y) * x) = 0.$ Therefore, $(X;*,0)$ is a BCL-algebra. □

In the next part we find a sufficient condition that makes a BCL-algebra be a d-algebra.

**Theorem 3.2:** A BCL-algebra $(X;*,0)$ satisfying $0 * x = 0$ for any $x \in X$ is a d-algebra.

Proof: The proof follows immediately from Definition 2.1. □

We will apply Theorem 3.2 to the next example.

**Example 3.2:** Consider the BCL-algebra $(X;*,0)$ given in [1, Theorem 2.4] with the following table:

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 3 | 1 |
| 2 | 2 | 3 | 0 | 2 |
| 3 | 3 | 0 | 0 | 0 |

It is obvious that d-1, d-2 and d-3 are applied in this example. Then $(X;*,0)$ is a BCL-algebra, which is a d-algebra.

**Theorem 3.3:** (See [1, Theorem 2.6]) If $(X;*,0)$ a BCL-algebra then the following relations are satisfied for any $x, y, z \in X$,

1) $(x*(x*y))*y = 0;$
2) $x*0=0$ imply $x=0$.

**Theorem 3.4:** (See [1, Theorem 2.1]) 1)Any BCK-algebra is a BCL-algebra;
2) Any BCI-algebra is a BCL-algebra;
3) Any BCH-algebra is a BCL-algebra.

Motivated by Theorem 2.1 in [1] (stated above in Theorem 3.4) we give our theorem which will show the sufficient conditions that we apply on BCL-algebra to become BCK/BCI/BCH respectively. Note that the last case were studied in [1] and the related theorem is given below.

**Theorem 3.5:** Let $(X;*,0)$ be a BCL-algebra. If $0*x = 0$ and $x*y = x*z$ for any $x, y, z \in X$, then

1) the BCL-algebra is a BCK-algebra;
2) the BCL-algebra is a BCI-algebra.

Proof: It is clear that the axioms BCK-1, BCK-2, BCK-3 are satisfied. With the assumptions given above, we have $((x*y)*(x*z))*(z*y) = 0*(z*y) = 0$. This proves that the axiom BCK-4 is valid. Finally, we know from Theorem 3.3 that a BCL-algebra satisfies the relation BCK-5. Thus the BCL-algebra is a BCK-algebra.

Similarly, we can observe that the axioms BCI-1 and BCI-3 follows directly from Definition 2.1. Also, BCI-2 and BCI-5 are valid from Theorem 3.3. We show that BCI-4 is valid using the assumptions above as we done in the first part. This proves that the given BCL-algebra is a BCI-algebra. □

We remind the reader that d-$4^+$ is the axiom $(x*y)*z = (x*z)*y$.

**Theorem 3.6:** (See [1, Theorem 2.2] ) If $(X;*,0)$ is a BCL-algebra satisfying d-$4^+$ then the BCL-algebra is a BCH-algebra.

**Corollary 3.1:** Any d-algebra $(X;*,0)$ satisfying d-$4^+$ is a BCH-algebra.

Proof: It is clear that BCH-1, BCH-2 are satisfied in any d-algebra and BCH-3 is d-$4^+$. Hence any d-algebra satisfying d-$4^+$ is a BCH-algebra. □

## 4. DEFORMATION OF BCL-ALGEBRA

In this section we study deformation of BCL-algebra. We start with basic definitions taken from [3].

**Definition 4.1:** Let $(X;*,0)$ be an algebra. A map $\varphi : X \to X$ is said to be a deformation function of $X$ if

(i) $x \neq 0$ implies $x*\varphi(x) \neq 0,$
(ii) there exist $a \in X$ such that $a*\varphi(a) \neq a.$

The element $a$ is called a deformation point of $X$ and $(X;*,0)$ is said to be a deformation algebra.

Next we will apply the notions in Definition 4.1 to a given BCL-algebra.

**Example 4.1:** Consider the algebra given in Example 3.2. Define a map $\varphi$ by $\varphi(0) = \varphi(1) = 0, \varphi(2) = 1, \varphi(3) = 0$. Then we have $1 * \varphi(1) = 1 * 0 = 1 \neq 0$. Similarly we can see that $2 * \varphi(2) \neq 0$ and $3 * \varphi(3) \neq 0$. Furthermore, there exists $2 \in X$ such that $2 * \varphi(2) \neq 2$. Therefore, the map $\varphi$ is a deformation function, the element 2 is a deformation point of $X$ and $(X;*,0)$ is a deformation algebra.

**Proposition 4.1:** Let $(X;*,0)$ be a BCL- algebra with $0 * x = 0$ and let $\varphi$ be a deformation function of $X$. Define a binary operation on $X$ by:

$x \nabla y := (x * y) * \varphi(x * y)$ for any $x, y \in X$, then $(X; \nabla, 0)$ is a d-algebra which is no t a BCL-algebra.

Proof: Given $(X;*,0)$ is a BCL- algebra and that $0 * x = 0$, by using the axioms in Definition 2.1 we have

$x \nabla x = (x * x) * \varphi(x * x) = 0 * \varphi(0) = 0$. Also $0 \nabla x = (0 * x) * \varphi(0 * x) = 0 * \varphi(0) = 0$. Assume that $x \nabla y = 0 = y \nabla x$. Then $(x * y) * \varphi(x * y) = 0 = (y * x) * \varphi(y * x)$. As $\varphi$ is a deformation function we get $x * y = 0 = y * x$. Hence, $x = y$. Therefore, $(X; \nabla, 0)$ is a d-algebra. We show that $(X; \nabla, 0)$ is not a BCL-algebra by providing the next example.□

**Example 4.2:** Consider the BCL-algebra given in Example 3.2 and consider the deformation function $\varphi$ given in Example 4.1. If we define $x \nabla y := (x * y) * \varphi(x * y)$ then $(X; \nabla, 0)$ is a deformed BCL-algebra (defined below) which is not BCL-algebra since $((1\nabla 3)\nabla 2)\nabla((1\nabla 2)\nabla 3)\nabla((2\nabla 3)\nabla 1) = 3 \neq 0$.

| $\nabla$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 3 | 1 |
| 2 | 3 | 3 | 0 | 3 |
| 3 | 3 | 0 | 0 | 0 |

**Lemma 4.1:** If $x \nabla y = 0$ then $x * y = 0$ for any $x, y \in X$.

**Corollary 4.1:** Let $(X;*,0)$ be a BCH- algebra with $0 * x = 0$ and let $\varphi$ be a deformation function of $X$. Define a binary operation on $X$ by:

$x \nabla y := (x * y) * \varphi(x * y)$ for any $x, y \in X$, then $(X; \nabla, 0)$ is a d-algebra which is not a BCH-algebra.

**Proof:** The proof is the same as the proof of Proposition 4.1 above using Definition 2.3. To verify that $(X; \nabla, 0)$ is not a BCH-algebra, consider the algebra $(X;*,0)$ defined as follows:

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 2 | 1 | 0 |

It is not difficult to check that $(X;*,0)$ is a BCH-algebra. Define the deformation function as given in Example 4.1. Then it is clear from the following table tha t the algebra $(X;\nabla,0)$ is a d-algebra and it is easy to check that the axiom BCH-3 fails as $((1\nabla 3)\nabla 2)\nabla((1\nabla 2)\nabla 3)=1\neq 0.$ Hence, $(X;\nabla,0)$ is not a BCH-algebra.

| $\nabla$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 3 | 3 |
| 2 | 3 | 3 | 0 | 1 |
| 3 | 3 | 3 | 1 | 0 |

□

**Definition 4.2:** An algebra is said to be rigid if it has no non-trivial deformation.

**Example 4.3:** Consider the BCL-algebra given in Example 3.2 and define a deformation function as follows $\varphi(0)=0,\ \varphi(1)=2, \varphi(2)=1, \varphi(3)=0.$ With direct calculations we can show that the deformed algebra $(X;\nabla,0)$ is a BCL-algebra where $(X;\nabla,0)$ is defined below. Thus the BCL-algebra $(X;*,0)$ in this example is not rigid. Note that the algebra $(X;\nabla,0)$ is a d-algebra.

| $\nabla$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 3 | 0 | 3 | 3 |
| 2 | 3 | 3 | 0 | 3 |
| 3 | 3 | 0 | 0 | 0 |

**Definition 4.3:** An algebra $(X;*,0)$ is said to be divisible if for any non-zero element $x\in X,$ there exists an element $\hat{x}\in X$ such that $x*\hat{x}\notin\{0,x\}.$ The element $\hat{x}$ is called an associator of $x.$

**Example 4.4:** Consider the algebra in Example 3.2. We can see that 2 is an associator of 1 and 1 is an associator of 2. Whereas, 3 has no associator. Hence, the given algebra is not divisible.

**Remark 4.1:** The associator is not unique in general.

**Proposition 4.2:** There exist some BCL-algebras $(X;*,0)$ which are not divisible.

Proof: Let $(X;*,0)$ be a BCL-algebra then $x*x=0$ and for any $x,y\in X, x\neq 0,\ x*y\in X.$ Therefore, we might have the cases where $x*y=0$ or $x*y=x$ i.e. we might have $x*y\in\{0,x\}.$ If this is the case then there is no associator $\hat{x}$ in $X$ such that $x*\hat{x}\notin\{0,x\}.$ Hence $(X;*,0)$ is not always divisible. □

**Theorem 4.1:** Let $(X;*,0)$ be a divisible BCL- algebra and define for a non-zero element $a\in X,$ a map $\varphi_a : X\to X$ by

$$\varphi_a(x)=\begin{cases}\hat{a} & x=a\\ 0 & x\neq a.\end{cases}$$

Then $\varphi_a$ is a deformation function of $X.$

Proof: We will use the same strategy used in the proof of [3, Theorem 4.7].

Let $x \neq 0$ then

$$x * \varphi_a(x) = \begin{cases} a * \varphi_a(a) = a * \hat{a}, & x = a \\ x * \varphi_a(x) = x * 0, & x \neq a. \end{cases}$$

Thus, if $x = a$, we have $a * \varphi_a(a) = a * \hat{a} \notin \{0, a\}$ as $X$ is a divisible algebra. If $x \neq a$, given that $x \neq 0$, then from Definition 1.1 BCL-2 we see that $x * 0 \neq 0$. Hence, $x * \varphi_a(x) \neq 0$. This proves that $\varphi_a$ is a deformation function of $X$. $\square$

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